Boundary Behaviors at ∞ for Fragments in Simple Exchangeable Fragmentation-Coalescence Processes

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Outline of the Talk

Partitions of positive integers

- $[n] = \{1, \ldots, n\}$. $\mathbb{N} = \{1, 2, \ldots\}$.
- A partition $\pi = \{\pi_i, i = 1, 2, \ldots\}$ of $D \subset \mathbb{N}$ is a collection of disjoint subsets, called blocks, such that $\cup_i \pi_i = D$ and $\min \pi_i < \min \pi_j$ for $i < j$, i.e. π_is are ordered by their least elements.
- It is used to keep track of genealogy of a population.
- A random partition π on $[n]$ is exchangeable if for any permutation σ of $[n]$, $\sigma(\pi)$ has the same law as π.
- A random partition π on $[\infty]$ is exchangeable if and only if the restriction $\pi_{|[n]}$ is exchangeable for any $n \in \mathbb{N}$.

Fragmentation of partitions

- Given a partition π of $B \subset \mathbb{N}$ with $\#\pi = n$ blocks, let π^f be a partition on B.
- For $1 \leq j \leq n$, we can use π^f to "break" block π_j of partition π to form a new partition

$$\text{Frag}(\pi, \pi^f, j) = \{\pi_j \cap \pi_i^f, i \geq 1\} \cup \{\pi_k, k \neq j\}.$$

- For example, if $\pi = \{\{1,3\},\{2,4\}\}$, then

$$\text{Frag}(\pi, \{\{1,2,3,4\}\}, 1) = \pi$$

and

$$\text{Frag}(\pi, \{\{1,2\},\{3,4\}\}, 2) = \{\{1,3\},\{2\},\{4\}\}.$$

Coagulation of partitions

- Given a partition π on $B \subset \mathbb{N}$ with $\#\pi = n$ and partition π' on $[n]$, we can use π' to merge blocks from π.
  $\text{Coag}(\pi, \pi') = (\pi''_j)$ where

$$\pi''_j := \cup_{i \in \pi'_j} \pi_i.$$

- For example, $B = [10]$,

$$\pi = \{\{1,6,7\},\{2,4,5\},\{3,8\},\{9,10\}\},$$

then

$$\text{Coag}(\pi, \{\{1\},\{2,3,4\}\}) = \{\{1,6,7\},\{2,3,4,5,8,9,10\}\}$$

which is a Λ-coalescent event and

$$\text{Coag}(\pi, \{\{1,3\},\{2,4\}\}) = \{\{1,3,6,7,8\},\{2,4,5,9,10\}\}$$

which is Ξ-coalescent event.

Exchangeable fragmentation-coalescence (EFC) process

- An EFC (exchangeable fragmentation-coalescence) process $(\Pi(t), t \geq 0)$ is an exchangeable $\mathbb{N}$-partition-valued Markov process that undergoes random fragmentation and coagulation, which can be constructed using Poisson point process of coalescence and fragmentation.
- It was first proposed and studied by Berestycki (2004).
- We consider a simple version of EFC processes with fragmentation occurring at a finite rate, and only Λ-coalescent is allowed.

To be precise, we only consider the following coalescences and fragmentations.

- Λ-*coalescents*: given there are n blocks, any k of them for $2 \leq k \leq n$ merge together to form a new block at rate

$$\lambda_{n,k} := \int_{[0,1)} x^k (1-x)^{n-k} x^{-2} \Lambda(\mathrm{d}x)$$

  for a finite measure Λ on $[0,1)$.
- *Fragmentations*: each block independently splits into $k+1$ blocks at rate $\mu(k)$. We always assume that $\mu(\infty) = 0$. At the level of the number of blocks, the fragmentation is therefore a discrete branching process with no death, whose offspring measure is μ.

Coming down from infinity (CDI)

- A Λ-coalescent comes down from infinity if starting with infinitely many blocks, the number of blocks becomes finite as soon as $t > 0$. It stays infinite if it never becomes finite.
- If $\Lambda(1) = 0$, then the Λ-coalescent either comes down from infinity or stays infinite.
- The following necessary and sufficient condition for coming down from infinity of Λ-coalescents was discovered by Schweinsberg (2000). Define for any $n \geq 2$,
  $\Phi(n) := \sum_{k=2}^{n}(k-1)\binom{n}{k}\lambda_{n,k}$.
  The Λ-coalescent $(\Pi(t), t \geq 0)$ comes down from infinity if and only if

$$\sum_{n=2}^{\infty} \frac{1}{\Phi(n)} < \infty \qquad \text{(Schweinsberg's condition)}.$$

Explosion for branching processes

- Explosion occurs if a process reaches ∞ in finite time with a positive probability.
- Consider a pure birth branching process $(N_t, t \geq 0)$ with offspring measure μ.
  For any $n \geq 1$,
  $$\ell(n) := \sum_{k=1}^{n} \bar{\mu}(k), \tag{1}$$
  where for any $k \in \mathbb{N}$, $\bar{\mu}(k) := \mu(\{k, k+1, ...\})$. The process $(N_t, t \geq 0)$ explodes if and only if
  $$\sum_{n=1}^{\infty} \frac{1}{n\ell(n)} < \infty \quad \text{(Doney's condition)}. \tag{2}$$

Previous related work

The behaviors at ∞ of EFC processes, had been studied in Kyprianou et al.(2017), where the "fast" fragmentation-coalescence process, in which

- the coagulation is a binary Kingman coalescent, a special case of Λ-coalescent,
- and in the fragmentation the block splits into infinitely many blocks.

A phase transition is found between a regime for which the boundary is an *exit* and a regime where the boundary ∞ is *regular*. Detailed computations are carried out.

The block counting process as a Markov chain

The generator of $(\#\Pi(t), t \geq 0)$ acts on functions on $\mathbb{N}$ as follows:

$$\mathcal{L}g := \mathcal{L}^c g + \mathcal{L}^f g \tag{3}$$

with coalescence generator

$$\mathcal{L}^c g(n) := \sum_{k=2}^{n} \binom{n}{k} \lambda_{n,k}[g(n-k+1) - g(n)]$$

and fragmentation generator

$$\mathcal{L}^f g(n) := n \sum_{k=1}^{\infty} \mu(k)[g(n+k) - g(n)]$$

for $n \in \mathbb{N}$, where $\mathcal{L}^c g(n)$ vanishes if $n = 1$.

The process goes up like a pure birth branching process and goes down like a Λ-coalescent.

Motivation of this work.

Boundary behaviors for Markov chain

Let N be a continuous time Markkov chain taking values in $\mathbb{N} \cup \{\infty\}$. We recall that

- the boundary ∞ is said to be an exit if the process $(N_t, t \geq 0)$ can reach ∞ but can not leave from ∞;
- the boundary ∞ is said to be an entrance if the process N can leave from ∞ but can not reach ∞;
- the boundary ∞ is called regular if the process N can both enter ∞ and leave from ∞.

CDI/Stay-Infinite for the block counting process

Proposition (Foucart AAP2021+)

If $\Phi(n) \underset{n\to\infty}{\sim} dn^{\beta+1}, \beta \in (0,1]$ *and* $\mu(n) \underset{n\to\infty}{\sim} \frac{b}{n^{\alpha+1}}$ *with* $\alpha \in (0,1)$ *and* $b, d > 0$, *let*

$$\theta = \limsup_{n\to\infty} \sum_{k=1}^{\infty} \frac{n\bar{\mu}(k)}{\Phi(n+k)} = \liminf_{n\to\infty} \sum_{k=1}^{\infty} \frac{n\bar{\mu}(k)}{\Phi(n+k)}.$$

- *If* $\alpha + \beta < 1$, *then* $\theta = \infty$ *and* $(\#\Pi(t), t \geq 0)$ *stays infinite;*
- *If* $\alpha + \beta > 1$, *then* $\theta = 0$ *and* $(\#\Pi(t), t \geq 0)$ *comes down from infinity;*
- *If* $\alpha + \beta = 1$, *then* $\theta = \frac{b}{d}\frac{1}{\alpha(1-\alpha)} \in (0, \infty)$, *and further,*
  - *if* $\frac{b}{d}\frac{1}{\alpha(1-\alpha)} > 1$, *then* $(\#\Pi(t), t \geq 0)$ *stays infinite;*
  - *if* $\frac{b}{d}\frac{1}{\alpha(1-\alpha)} < 1$, *then* $(\#\Pi(t), t \geq 0)$ *comes down from infinity.*

Non-explosion for Markov chain

Consider a general continuous time Markov chain $(N_t, t \geq 0)$ on $\mathbb{N}$ with infinitesimal generator $\mathscr{L} = \mathscr{L}^- + \mathscr{L}^+$ acting on all bounded function $g : \mathbb{N} \to \mathbb{R}_+$ and all $n \in \mathbb{N}$ as follows.

$\mathscr{L}^- g(n) = \sum_{k=1}^{n-1} \left(g(n-k) - g(n)\right) p^-_{n,k}$,

$\mathscr{L}^+ g(n) = \sum_{k=1}^{\infty} \left(g(n+k) - g(n)\right) p^+_{n,k}$

where $p^+_{n,k} \in [0, \infty)$ and $p^-_{n,k} \in [0, \infty)$ are respectively the rates of positive and negative jumps.

We recall a classical Foster-Lyapunov criterion for non-explosion.

If there exists a non-decreasing function $f : n \mapsto f(n)$ such that $f(n) \underset{n\to\infty}{\longrightarrow} \infty$ and

$$\mathscr{L} f(n) \leq c f(n) \text{ for all } n \geq 1 \tag{4}$$

for some $c > 0$, then the (minimal) continuous-time Markov chain $(N_t, t \geq 0)$ does not explode.

Explosion for Markov chain

Using an approach in Li, Yang and Z. (2019) for showing explosion of SDE with positive jumps, we can prove the following result on explosion of the Markov chain.

For any $a > 0$ and for any $n \in \mathbb{N}$, set $g_a(n) := n^{1-a}$ and $G_a(n) := -\frac{1}{n^{1-a}} \mathscr{L} g_a(n)$.

Theorem (Foucart and Z. AIHP2021+)

If there exists an eventually non-decreasing positive function g satisfying $\int^{\infty} \frac{\mathrm{d}x}{xg(x)} < \infty$ such that for some $a > 1$,

$$G_a(n) \geq g(\log n) \log n \tag{5}$$

for all large enough n, then $\mathbb{P}_n(\tau_\infty^+ < \infty) > 0$ for all large enough $n \in \mathbb{N}$.

It agrees with Doney's condition.

Sufficient conditions for explosion (non-explosion)

$$G_a^-(n) := -\frac{1}{n^{1-a}}\mathscr{L}^- g_a(n), \quad G_a^+(n) := -\frac{1}{n^{1-a}}\mathscr{L}^+ g_a(n).$$

Proposition

Assume that there are $a > 1$ and a non-decreasing positive function g such that $\int^\infty \frac{\mathrm{d}x}{xg(x)} < \infty$, and for large enough n, $G_a^+(n) \geq g(\log n)\log n$. If $\gamma_a := \limsup_{n\to\infty} \frac{-G_a^-(n)}{G_a^+(n)} < 1$, then Condition (5) holds and explosion occurs for any large initial state.

Proposition

If there exists $a < 1$ such that $\mathscr{L}^+ g_a(n) < \infty$ for all $n \in \mathbb{N}$, and $\limsup_{n\to\infty} \frac{-G_a^+(n)}{G_a^-(n)} < 1$, then $\mathscr{L} g_a(n) \leq 0$ for large enough n and Condition (4) holds with $f = g_a$. The Markov chain does not explode.

Boundary classification at ∞

Combining results of CDI/stay-infinite and explosion/non-explosion

Theorem

(Foucart and Z. 2021+) Assume that $\Phi(n) \underset{n\to\infty}{\sim} dn^{1+\beta}$ *with* $d > 0$ *and* $\beta \in (0,1)$ *and* $\mu(n) \underset{n\to\infty}{\sim} \frac{b}{n^{\alpha+1}}$ *with* $b > 0$ *and* $\alpha \in (0,\infty)$. *Then*

- *if* $\alpha + \beta < 1$, *then* ∞ *is an exit boundary,*
- *if* $\alpha + \beta > 1$, *then* ∞ *is an entrance boundary,*
- *if* $\alpha + \beta = 1$ *and further,*
  - *if* $b/d > \alpha(1-\alpha)$, *then* ∞ *is an exit boundary,*
  - *if* $\frac{\alpha \sin(\pi\alpha)}{\pi} < b/d < \alpha(1-\alpha)$, *then* ∞ *is a regular boundary,*
  - *if* $b/d < \frac{\alpha \sin(\pi\alpha)}{\pi}$, *then* ∞ *is an entrance boundary.*

Phase transition at boundary ∞ for case $\alpha + \beta = 1$

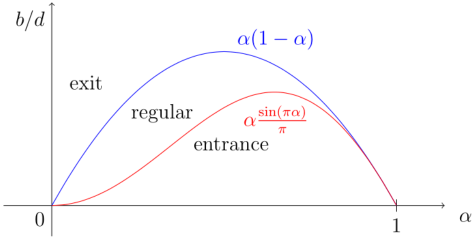

Figure: Boundary classification when $\Phi(n) \underset{n\to\infty}{\sim} dn^{2-\alpha}$ and $\mu(n) \underset{n\to\infty}{\sim} \frac{b}{n^{1+\alpha}}$.

Result for a critical case

Proposition

Given

$$\alpha+\beta=1, \Lambda(\mathrm{d}x)=d\frac{\beta(\beta+1)}{\Gamma(1-\beta)}x^{-\beta}\mathrm{d}x \quad \textit{and } \mu(n)=\frac{b}{n^{1+\alpha}}$$

for $b, d>0$, $\alpha\in(0,1)$ *and all integer* $n\geq 1$. *If* $\frac{b}{d}=\frac{\alpha\sin(\pi\alpha)}{\pi}$, *then the boundary* ∞ *is an entrance boundary.*

∞ regular for itself

The next proposition describes more precisely the behavior of the process $(\Pi(t), t \geq 0)$, with regularly varying coalescence-splitting measures, when the block-counting process has ∞ as regular boundary. We establish that the boundary ∞ is regular for itself, i.e. the block counting process returns to ∞ immediately after leaving from it.

Proposition

Suppose that the assumptions of Theorem 2 hold. If $\alpha + \beta = 1$ and $\frac{\alpha \sin(\pi\alpha)}{\pi} < b/d < \alpha(1-\alpha)$, then the process $(\Pi(t), t \geq 0)$ started from a partition with infinitely many blocks comes down from infinity and returns instantaneously to a proper partition with infinitely many blocks a.s.

Comments on the main proof

- The results for CDI/stay-infinite were obtained in Foucart (2021+) with the phase transition depending on $\frac{b}{d\alpha(1-\alpha)}$.
- We apply Propositions to show explosion/non-explosion results. For this purpose, we stop the process at the following first coalescence time when the number of blocks decreases by a proportion larger than p.

$$\sigma_p := \inf\{t \geq 0; \#\Pi(t) \leq (1-p)\#\Pi(t-)\}.$$

- By letting $p \to 0+$ and $a \to 1$, we can show the phase transition of explosion/non-explosion depends on $\frac{b}{d}\int_0^\infty \frac{\log(1+u)}{u^{1+\alpha}}\mathrm{d}u$.
- We need to compare $\int_0^\infty \frac{\log(1+u)}{u^{1+\alpha}}\mathrm{d}u$ and $\frac{1}{\alpha(1-\alpha)}$ for $0 < \alpha < 1$ to see whether ∞ can ever be a regular boundary.

Notice that

$$\begin{aligned}\int_0^\infty \frac{\log(1+u)}{u^{1+\alpha}}\mathrm{d}u &= \frac{1}{\alpha}\int_0^\infty \frac{1}{1+u}u^{-\alpha}\mathrm{d}u \\ &= \frac{1}{\alpha}\int_0^1 \frac{1}{v}\left(\frac{1}{v}-1\right)^{-\alpha}\mathrm{d}v \\ &= \frac{1}{\alpha}\int_0^1 v^{\alpha-1}(1-v)^{-\alpha}\mathrm{d}v \\ &= \frac{1}{\alpha}\mathrm{Beta}(\alpha,1-\alpha) = \frac{1}{\alpha}\Gamma(1-\alpha)\Gamma(\alpha) \\ &= \frac{1}{\alpha}\frac{\pi}{\sin(\pi\alpha)} > \frac{1}{\alpha(1-\alpha)}\end{aligned}$$

where we use Euler's reflection formula for gamma function.

Thank you for your attention!